Derivation of the Rotne-Prager mobility matrix

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1 Single particle centered at $\mathbf{r} = \mathbf{0}$

Step 1. Velocity field $\mathbf{v}_0(\mathbf{r})$ generated by a point-force $\mathbf{F}\delta(\mathbf{r})$,

$$\mathbf{v_0}(\mathbf{r}) = \mathbf{T}(\mathbf{r}) \cdot \mathbf{F}, \tag{1}$$

where $\mathbf{T}$ is the Oseen tensor.
Step 2. Velocity field $\mathbf{v}(\mathbf{r})$ generated by a sphere of radius a moving under the force $\mathbf{F}$,

$$\mathbf{v}(\mathbf{r}) = \left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathbf{v}_0(\mathbf{r}). \tag{2}$$

Interpretation: the sphere is immersed in the ambient flow $\mathbf{v_0}(\mathbf{r})$.

2 Two spheres, sphere 1 under external force F

Step 3. Ambient velocity field, generated by sphere 1, with the center at $\mathbf{r} = \mathbf{0}$, incoming at the sphere 2 centered at $\mathbf{r} = \mathbf{r}_{12}$,

$$\mathbf{v}(\mathbf{r}_{12}) = \left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathbf{v}_0(\mathbf{r}_{12}). \tag{3}$$

Step 4. Faxen law: velocity of the center of sphere 2 immersed in external flow $\mathbf{v}(\mathbf{r}_{12})$,

$$\mathbf{U}_2 = \left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathbf{v}(\mathbf{r}_{12}). \tag{4}$$

Step 5. The Rotne-Prager mobility matrix $\boldsymbol{\mu}_{21}$ is defined by the following equation,

$$\mathbf{U}_2 = \boldsymbol{\mu}_{21} \cdot \mathbf{F}. \tag{5}$$

From Eqs. (1)-(4) it follows that

$$\boldsymbol{\mu}_{21} = \left(1 + 2\frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathbf{T}(\mathbf{r}_{12}). \tag{6}$$

because if $\mathbf{v}$ satisfies the Stokes equations, then $\boldsymbol{\nabla}^2\boldsymbol{\nabla}^2\mathbf{v} = \mathbf{0}$.

Warning. $\mathbf{U}_2$ is only an approximation, and it is not equal to velocity of particle 2 in the two-sphere system (the stick boundary conditions are not satisfied at sphere 1).