Stokes flow generated by a point force in various geometries
II. Velocity field

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The problem

The system consists of a point force in a viscous fluid in the vicinity of plane interface (free surface or rigid wall). The goal is to show the velocity field of the fluid, as well as the streamlines of the flow. The flow is assumed to be incompressible and satisfy the Stokes equations.

Geometry

Point force in a viscous fluid:

- Unbounded
- With a free boundary
  - Parallel to the force
  - Perpendicular to the force
- With a rigid wall
  - Parallel to the force
  - Perpendicular to the force

Normalization

- The used units:
  - Length scale: the wall-particle distance h.
  - Velocity $v_0 = \frac{F}{8\pi\mu h}$, where F is the absolute value of the force, μ is the viscosity coefficient.
- Dimensionless quantities:
  - Distance $\mathbf{r} = \frac{\tilde{r}}{h}$
  - Velocity $v(\mathbf{v}) = \frac{\tilde{v}}{v_0}$
    where $\tilde{r}, \tilde{v}$ are denormalized quantities
- The range of coordinates in every plot is the same and equal:

$$x \in (-3.14, 3.14) \qquad y \in (-3.14, 3.14)$$

Notation

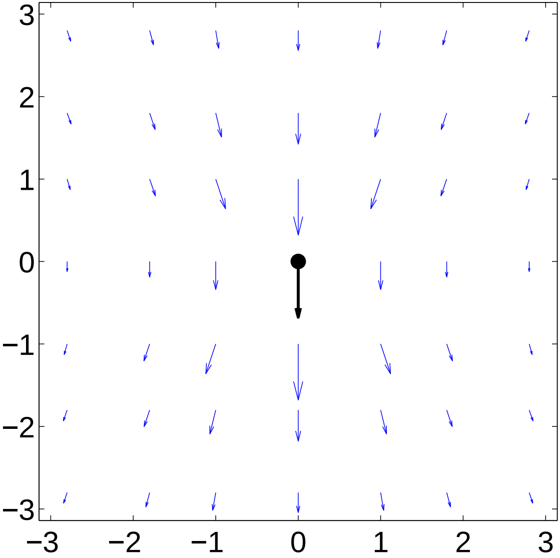
- The plots and equations are shown in the xy plane (at $z = 0$) which is perpendicular to the wall and parallel to the force.
- The relative positions are denoted as:
  - from the point force: $\mathbf{r} = [r_x, r_y]$,
  - from its image: $\mathbf{R} = [R_x, R_y]$.
- The x axis is always perpendicular to the force, while the y axis is always parallel to the force, with the opposite sense: $\mathbf{F} = (0, -F, 0)$ with $F > 0$.
- Color scale on all contour plots is the same and the representation is:
  - red for higher velocity values
  - blue for lower velocity values
- The point forces are black and their images are gray.
- The free surface is a dashed black line, the rigid wall is a solid black line.
- The length of a vector on velocity field plot represents the magnitude of velocity. The vectors are scaled so they do not overlap, relative to the longest vector on the sequence of plots.

Unbounded fluid

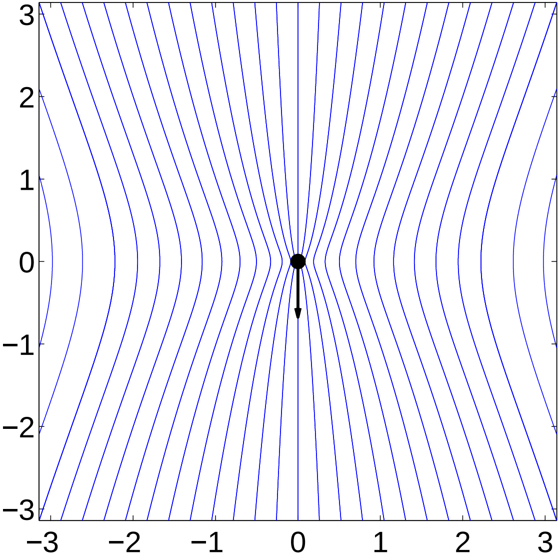
The analysis of a problem of Stokes flow in an unbounded fluid (boundary condition of velocity vanishing at infinity) brings up a mathematical formula for velocity in any point in space [1]. The formulae for components of the normalized velocity vector $[u_x, u_y]$ for point force at $(0, 0, 0)$ are:

$$\begin{aligned} u_x &= -\frac{r_x r_y}{r^3}, \\ u_y &= -\frac{1}{r} - \frac{r_y^2}{r^3}, \end{aligned}$$

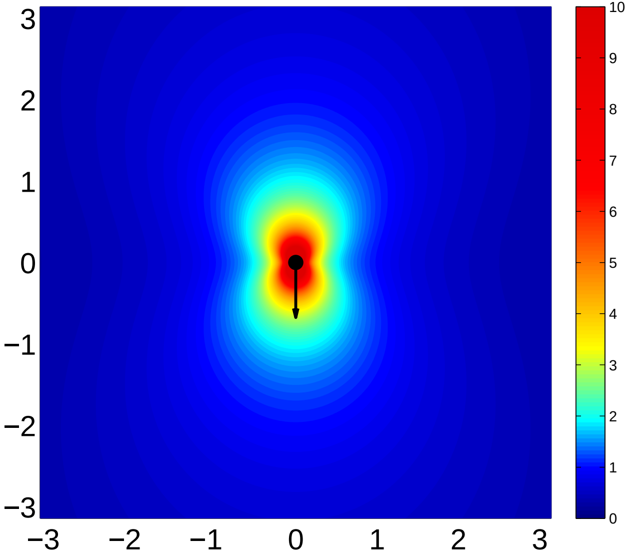
Unbounded fluid - velocity field



Unbounded fluid - streamlines



Unbounded fluid - velocity isolines (lines of constant velocity value)



Free surface - equations

The boundary condition on a free surface (normal velocity vanishes and stress parallel to the surface is equal to zero) is satisfied by adding a velocity field of a particle image placed on the other side of the surface. That method is called the method of images [2]. The velocity vector components $\tilde{u_x}$ and $\tilde{u_y}$ in case of a free surface positioned at $x = 0$, parallel to the point force at $(h, 0, 0)$, are equal to:

$$\begin{aligned}
\tilde{u}_x &= -\frac{F}{8\pi\mu}\left(\frac{\tilde{r}_x\tilde{r}_y}{\tilde{r}^3} + \frac{\tilde{R}_x\tilde{R}_y}{\tilde{R}^3}\right), \\
\tilde{u}_y &= -\frac{F}{8\pi\mu}\left(\frac{1}{\tilde{r}} + \frac{\tilde{r}_y^2}{\tilde{r}^3} + \frac{1}{\tilde{R}} + \frac{\tilde{R}_y^2}{\tilde{R}^3}\right),
\end{aligned}$$

where $\tilde{r}_x = x - h$, $\tilde{R}_x = x + h$, $\tilde{r}_y = \tilde{R}_y = y$. In case of a point force at $(0, -h, 0)$, perpendicular to the free surface positioned at $y = 0$, the formulae are:

$$\begin{aligned}
\tilde{u}_x &= \frac{F}{8\pi\mu}\left(-\frac{\tilde{r}_x\tilde{r}_y}{\tilde{r}^3} + \frac{\tilde{R}_x\tilde{R}_y}{\tilde{R}^3}\right), \\
\tilde{u}_y &= \frac{F}{8\pi\mu}\left(-\frac{1}{\tilde{r}} - \frac{\tilde{r}_y^2}{\tilde{r}^3} + \frac{1}{\tilde{R}} + \frac{\tilde{R}_y^2}{\tilde{R}^3}\right),
\end{aligned}$$

where $\tilde{r}_x = \tilde{R}_x = x$, $\tilde{r}_y = y + h$, $\tilde{R}_y = y - h$.

Free surface - normalized equations

The velocity vector components u_x and u_y in case of a free surface positioned at $x = 0$, parallel to the point force at $(1, 0, 0)$, are equal to:

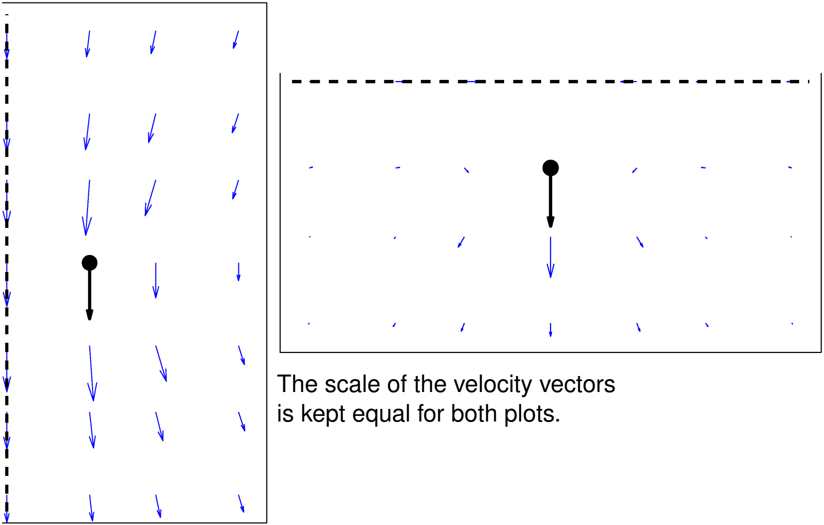
$$u_x = -\frac{r_x r_y}{r^3} - \frac{R_x R_y}{R^3},$$
$$u_y = \left(-\frac{1}{r} - \frac{r_y^2}{r^3}\right) + \left(-\frac{1}{R} - \frac{R_y^2}{R^3}\right),$$

where $r_x = x - 1$, $R_x = x + 1$, $r_y = R_y = y$. In case of a point force at $(0, -1, 0)$, perpendicular to the free surface positioned at $y = 0$, the formulae are:

$$u_x = -\frac{r_x r_y}{r^3} + \frac{R_x R_y}{R^3},$$
$$u_y = \left(-\frac{1}{r} - \frac{r_y^2}{r^3}\right) + \left(\frac{1}{R} + \frac{R_y^2}{R^3}\right),$$

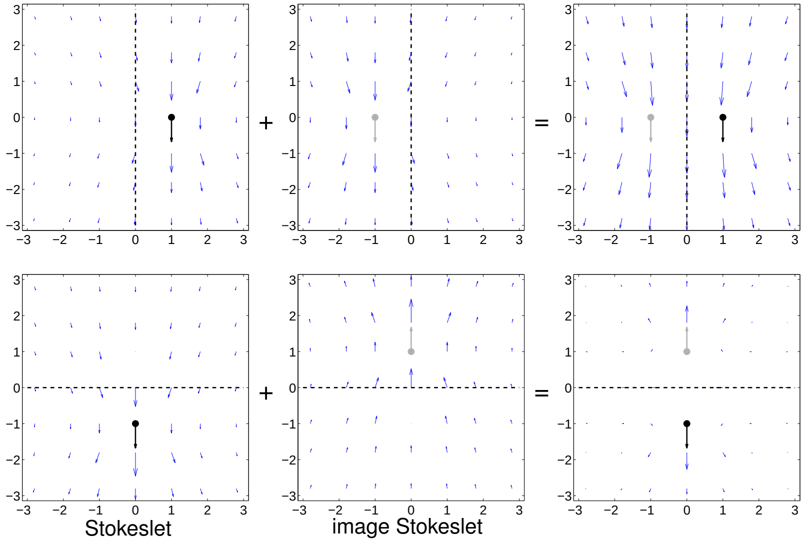
where $r_x = R_x = x$, $r_y = y + 1$, $R_y = y - 1$.

Free surface parallel and perpendicular to the force. Velocity field.

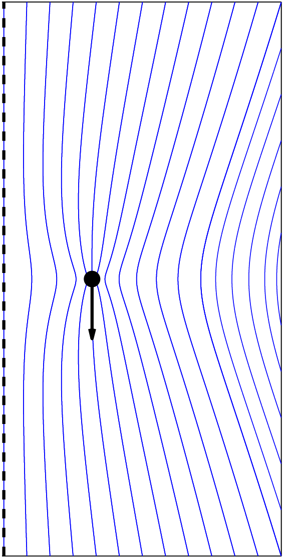


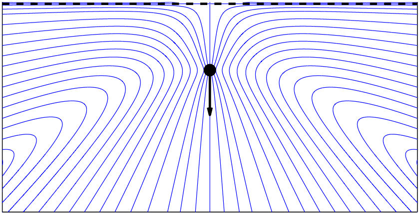
The scale of the velocity vectors is kept equal for both plots.

Free surface parallel and perpendicular to the force. Velocity field.

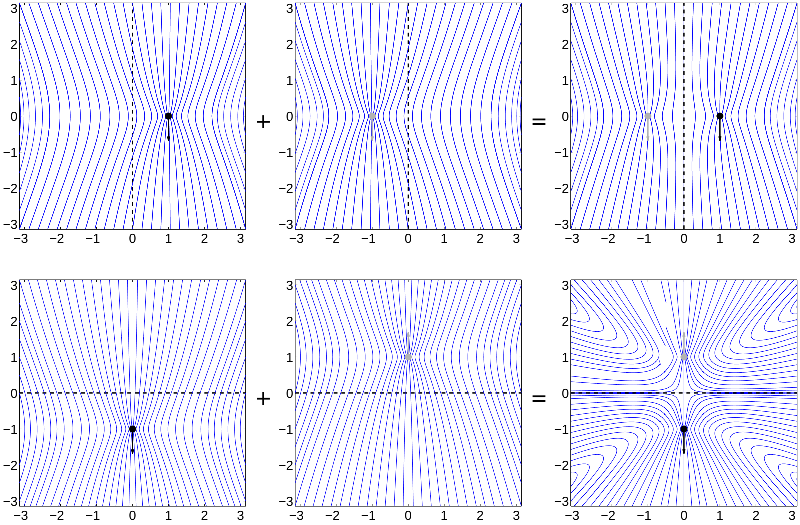


Free surface parallel and perpendicular to the force. Streamlines.

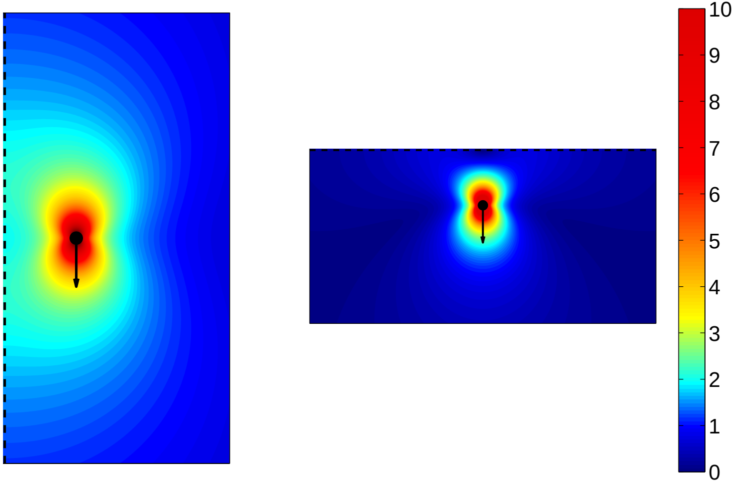




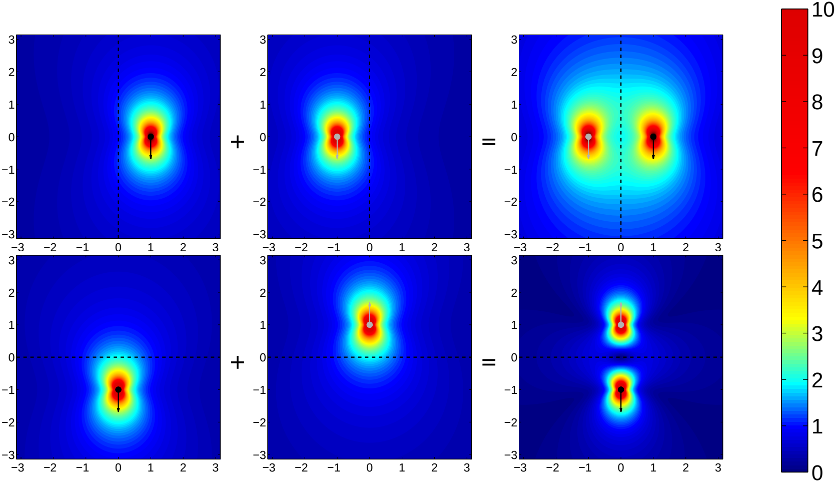
Free surface parallel and perpendicular to the force. Streamlines.



Free surface parallel and perpendicular to the force. Velocity isolines.



Free surface parallel and perpendicular to the force. Velocity isolines.



Rigid wall parallel to the force - equations

The boundary condition on a rigid wall is that the velocity of the fluid is equal to the velocity of the wall (so called no-slip condition). In case of a stationary wall the velocity has to be zero. This condition is satisfied using the method of images [3]. The formulae for velocity vector components $\tilde{u}_x$ and $\tilde{u}_y$ in case of a rigid wall at $x = 0$ in vicinity of a point force at $(h, 0, 0)$ parallel to the wall are:

$$\tilde{u}_x = \frac{-F}{8\pi\mu}\left(\underbrace{\frac{\tilde{r}_x\tilde{r}_y}{\tilde{r}^3}}_{\text{Stokeslet}} \underbrace{-\frac{\tilde{R}_x\tilde{R}_y}{\tilde{R}^3}}_{\text{image Stokeslet}} \underbrace{+\frac{6h\tilde{R}_x^2\tilde{R}_y}{\tilde{R}^5} + \frac{2h\tilde{R}_y}{\tilde{R}^3}}_{\text{image Stokeslet doublet}} \underbrace{-\frac{6h^2\tilde{R}_x\tilde{R}_y}{\tilde{R}^5}}_{\text{image source doublet}}\right),$$

$$\tilde{u}_y = \frac{-F}{8\pi\mu}\left(\underbrace{\frac{1}{\tilde{r}} + \frac{\tilde{r}_y^2}{\tilde{r}^3}}_{\text{Stokeslet}} \underbrace{-\frac{1}{\tilde{R}} - \frac{\tilde{R}_y^2}{\tilde{R}^3}}_{\text{image Stokeslet}} \underbrace{+\frac{6h\tilde{R}_x\tilde{R}_y^2}{\tilde{R}^5} - \frac{2h\tilde{R}_x}{\tilde{R}^3}}_{\text{image Stokeslet doublet}} \underbrace{+\frac{2h^2}{\tilde{R}^3} - \frac{6h^2\tilde{R}_y^2}{\tilde{R}^5}}_{\text{image source doublet}}\right),$$

where $\tilde{r}_x = x - h$, $\tilde{R}_x = x + h$, $\tilde{r}_y = \tilde{R}_y = y$.

Rigid wall perpendicular to the force - equations

In case of a point force at $(0,-h,0)$ perpendicular to the wall at $y=0$ the formulae are:

$$
\begin{aligned}
\tilde{u}_x &= \frac{-F}{8\pi\mu}\left(\underbrace{\frac{\tilde{r}_x\tilde{r}_y}{\tilde{r}^3}}_{\text{Stokeslet}}\underbrace{-\frac{\tilde{R}_x\tilde{R}_y}{\tilde{R}^3}}_{\text{image Stokeslet}}\underbrace{+\frac{2h\tilde{R}_x}{\tilde{R}^3}-\frac{6h\tilde{R}_x\tilde{R}_y^2}{\tilde{R}^5}}_{\text{image Stokeslet doublet}}\underbrace{+\frac{6h^2\tilde{R}_x\tilde{R}_y}{\tilde{R}^5}}_{\text{image source doublet}}\right), \\
\tilde{u}_y &= \frac{-F}{8\pi\mu}\left(\underbrace{\frac{1}{\tilde{r}}+\frac{\tilde{r}_y^2}{\tilde{r}^3}}_{\text{Stokeslet}}\underbrace{-\frac{1}{\tilde{R}}-\frac{\tilde{R}_y^2}{\tilde{R}^3}}_{\text{image Stokeslet}}\underbrace{+\frac{2h\tilde{R}_y}{\tilde{R}^3}-\frac{6h\tilde{R}_y^3}{\tilde{R}^5}}_{\text{image Stokeslet doublet}}\underbrace{+\frac{6h^2\tilde{R}_y^2}{\tilde{R}^5}-\frac{2h^2}{\tilde{R}^3}}_{\text{image source doublet}}\right),
\end{aligned}
$$

where $\tilde{r}_x=\tilde{R}_x=x$, $\tilde{r}_y=y+h$, $\tilde{R}_y=y-h$.

Rigid wall - normalized equations

The formulae for dimensionless velocity vector components u_x and u_y in case of a rigid wall at $x = 0$ in vicinity of a point force at $(1, 0, 0)$ parallel to the wall are:

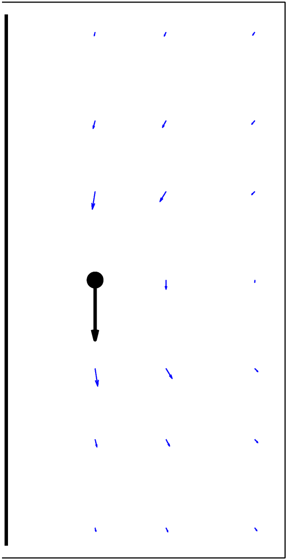
$$\begin{aligned} u_x &= \frac{r_x r_y}{r^3} - \frac{R_x R_y}{R^3} + \frac{2R_y}{R^3} + \frac{6R_x R_y (R_x - 1)}{R^5}, \\ u_y &= \frac{1}{r} - \frac{1}{R} + \frac{r_y^2}{r^3} - \frac{R_y^2}{R^3} + \frac{2(1 - R_x)}{R^3} + \frac{6{R_y}^2 (R_x - 1)}{R^5}, \end{aligned}$$

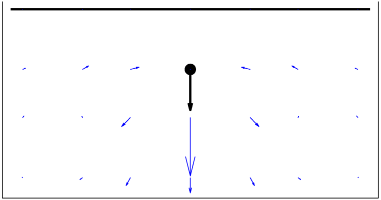
where $r_x = x - 1$, $R_x = x + 1$, $r_y = R_y = y$. In case of a point force at $(0, -1, 0)$ perpendicular to the wall at $y = 0$ the formulae are:

$$\begin{aligned} u_x &= \frac{r_x r_y}{r^3} - \frac{R_x R_y}{R^3} + \frac{2R_x}{R^3} + \frac{6R_x R_y (1 - R_y)}{R^5}, \\ u_y &= \frac{1}{r} - \frac{1}{R} + \frac{r_y^2}{r^3} - \frac{R_y^2}{R^3} + \frac{2(R_y - 1)}{R^3} + \frac{6{R_y}^2 (1 - R_y)}{R^5}, \end{aligned}$$

where $r_x = R_x = x$, $r_y = y + 1$, $R_y = y - 1$.

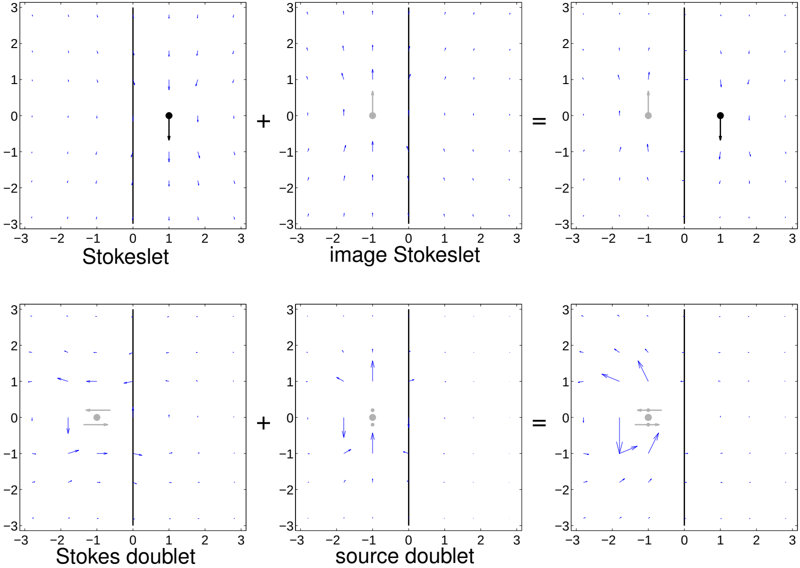
Rigid wall parallel and perpendicular to the force. Velocity field.



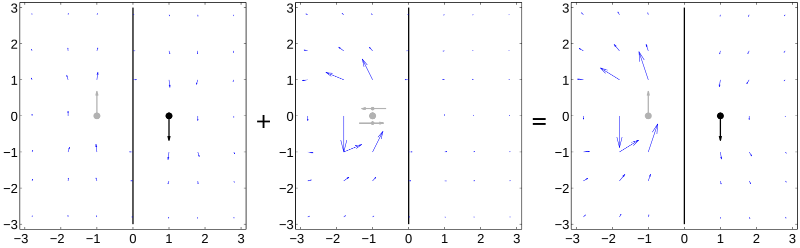


The scale of the velocity vectors is kept equal for both plots.

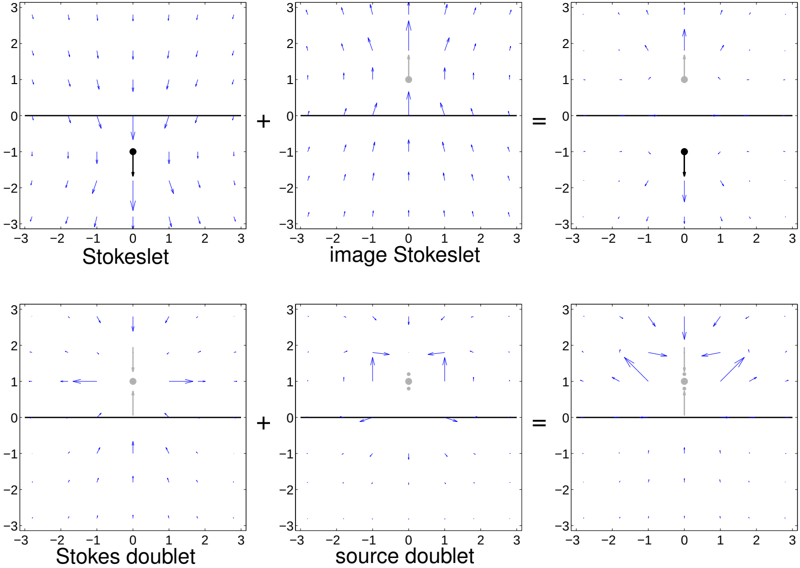
Rigid wall parallel to the force. Velocity field.



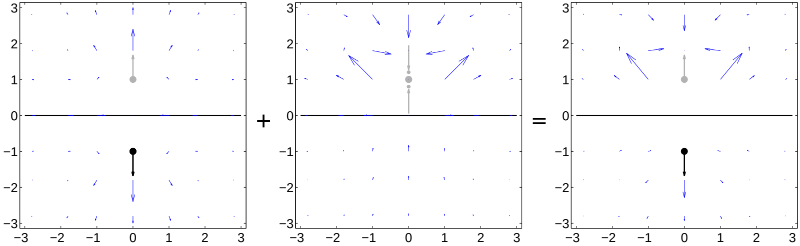
Rigid wall parallel to the force. Velocity field.



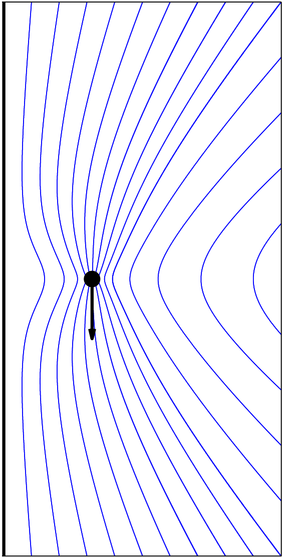
Rigid wall perpendicular to the force. Velocity field.

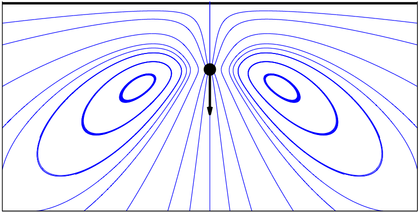


Rigid wall perpendicular to the force. Velocity field.

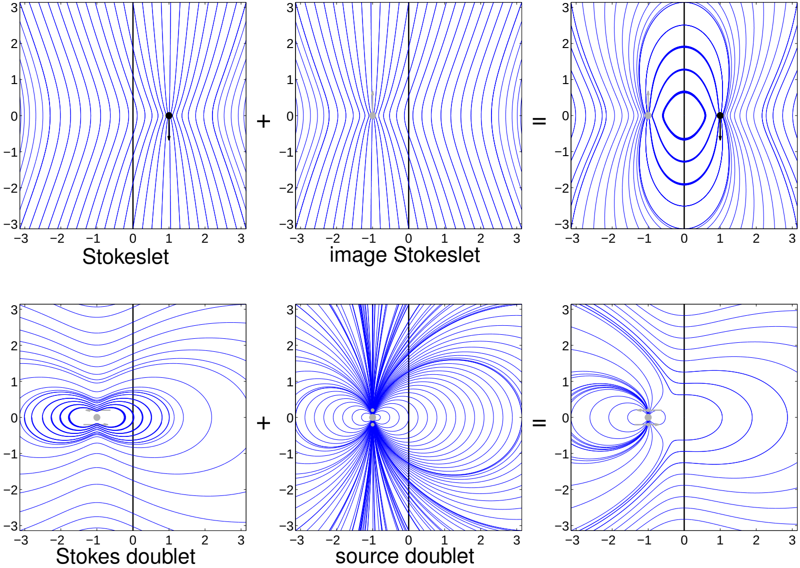


Rigid wall parallel and perpendicular to the force. Streamlines.

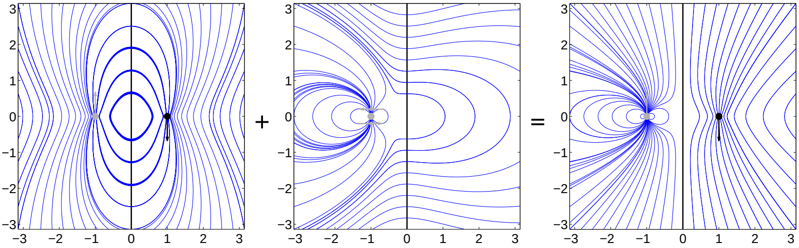




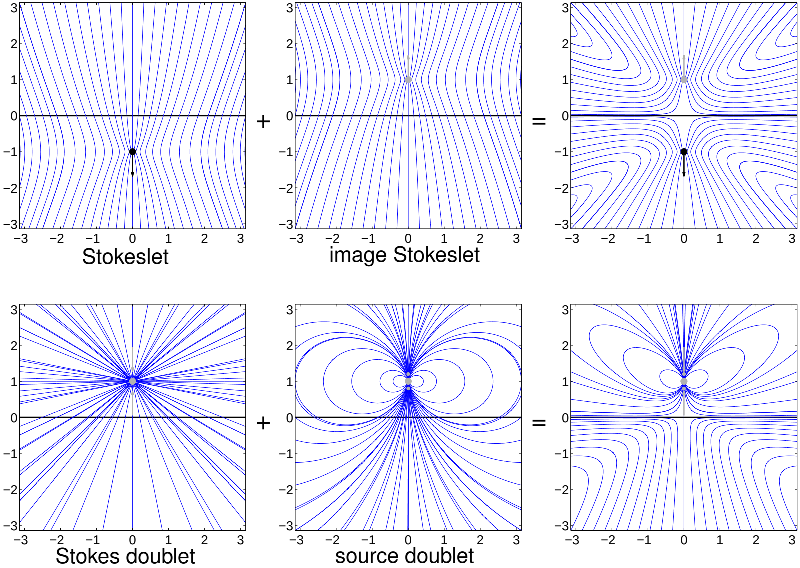
Rigid wall parallel to the force. Streamlines.



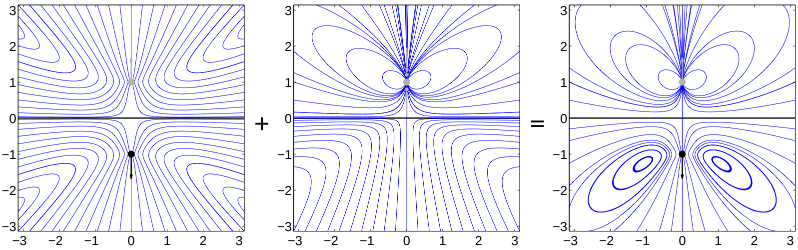
Rigid wall parallel to the force. Streamlines.



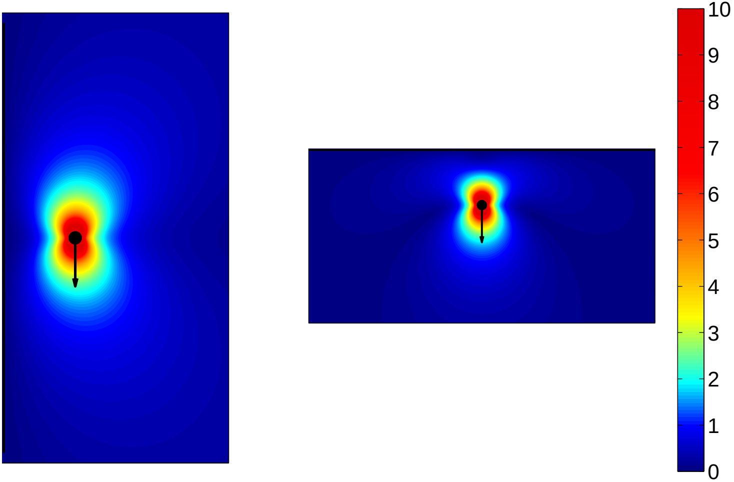
Rigid wall perpendicular to the force. Streamlines.



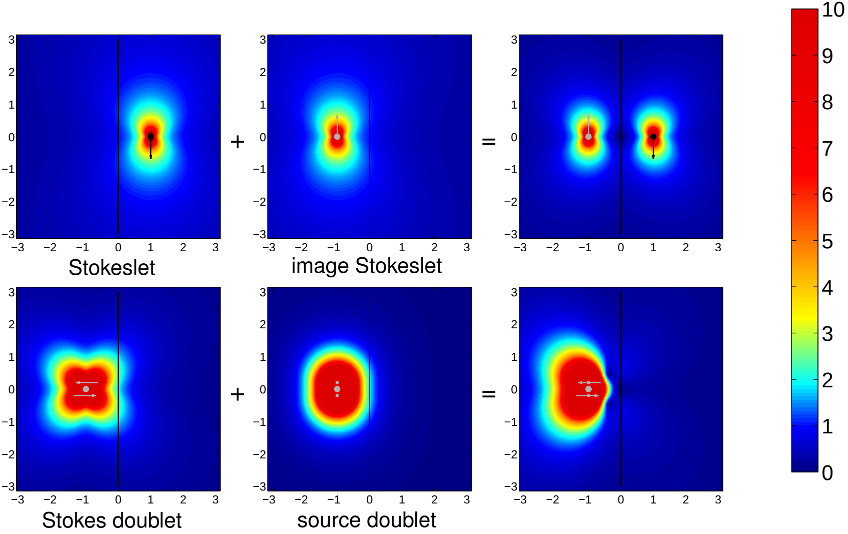
Rigid wall perpendicular to the force. Streamlines.



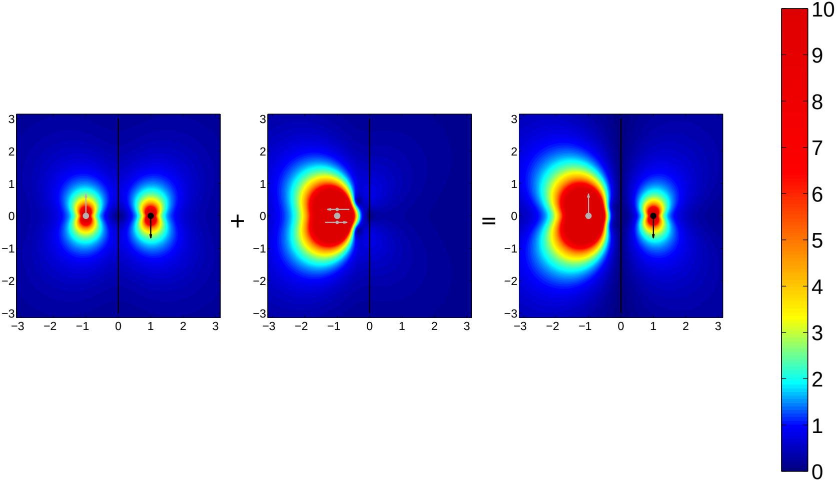
Rigid wall parallel and perpendicular to the force. Velocity isolines.



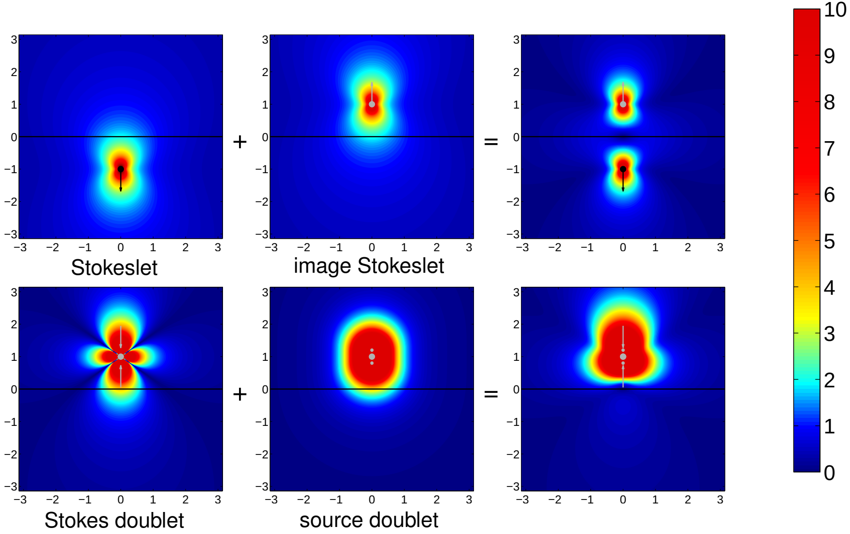
Rigid wall parallel to the force. Velocity isolines.



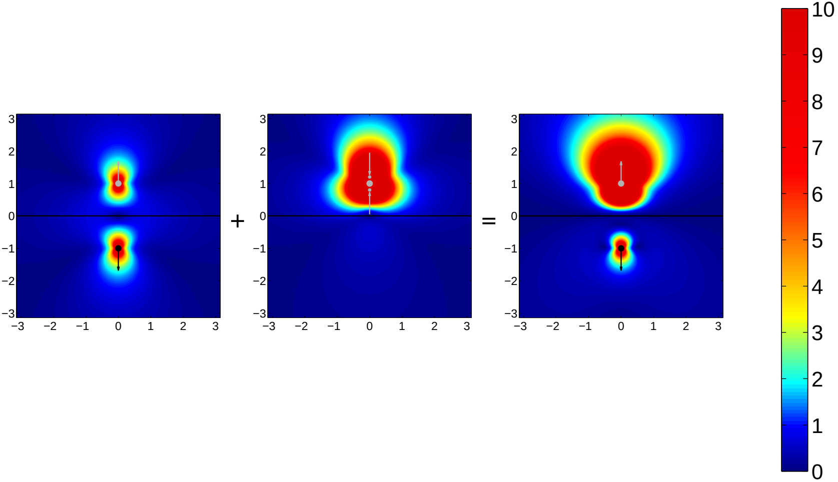
Rigid wall parallel to the force. Velocity isolines.



Rigid wall perpendicular to the force. Velocity isolines.



Rigid wall perpendicular to the force. Velocity isolines.



Literature

1. Kim S., Karilla S., *Microhydrodynamics. Principles and Selected Applications*, Dover, 2005.
2. G. S. Perkins, R. B. Jones. *Hydrodynamic interaction of a spherical particle with a planar boundary I. Free surface*, Physica A, **3**, 171 (1991)
3. J. R. Blake, *A note on the image system for a stokeslet in a no-slip boundary*, Proc. Camb. Phil. Soc., **70**, 303 (1971).